Prop 3.14 (cycl. rig. for inertia subgps [AbsTopIII, Prop1.4])

X gen ≥ 2 $\subset \overline{X}$

$\phi \neq U \underset{\text{open}}{\subset} X$ $x \in X(k) \setminus U(k)$ $U_x := \overline{X} \setminus \{x\}$

I_x : inertia at x $< \Delta_U$

(1). $\ker(\Delta_U \twoheadrightarrow \Delta_{U_x})$, $\ker(\Pi_U \twoheadrightarrow \Pi_{U_x})$ top. normally gen. by the inerti subgps of the pts of $U_x \setminus U$

(2). $1 \to I_x \to \Delta_{U_x}^{\text{cusp-cent}} \to \Delta_{\overline{X}} \to 1$

$\underset{\text{Leray}}{\leadsto}$ $E_2^{p,q} = H^p(\Delta_{\overline{X}}, H^q(I_x, I_x)) \Rightarrow H^{p+q}(\Delta_{U_x}^{\text{cusp-cent}}, I_x)$

$I_x, \Delta_{U_x}^{\text{cusp-cent}} \underset{\text{by Leray!}}{\cap} I_x$

$\sim \hat{\mathbb{Z}} = H_{\mathrm{sm}}(I_x,$

ell. cusp'tion $\leadsto$ • mono-theta env. const. mult. rig. $\leadsto$ { non-interference / cano. splitting

• recon. in Arch. theory (Aut-hol.)

• recon of NF ← gl. non-real'd & Kummer theory for gl. portion

Belyi cusp'tion

cycl. rig. $F^\times \cap \prod \mathcal{O}_v = \mu$ $\leadsto$ non-interfer-ce for "gl portion"

ring str. { $\boxtimes$-line hol / $\boxplus$-line lodp

concerning "val. gp portion" of Θ-link

we can protect val. gp portion from $\hat{\mathbb{Z}}^\times$-indet.

Θ-ll unit $\dagger\mathcal{O}^\times \cong {}^\ddagger\mathcal{O}^\times$ ($\hat{\mathbb{Z}}^\times$-indet)

• val. gp $\{\underline{q}^{j^2}\} \mapsto \underline{\underline{q}}$ (cusp)

• gl. real'd $\mathbb{R}_{\geq 0}$

op(.4])

s

sumally gen. by
the inerti. subgps
of the pts of $U_X \setminus U$

$I_x)) \to H^{n+q}(\Delta^{\text{cusp-cut}}_{U_X}, I_X)$

I_x

proof | omit //

Lem 3.15 ([AbsTopIII, Prop 1.6])

k: Kummer-faithful, X: proper

$\phi \neq U \underset{\text{open}}{\subset} X$

$\kappa_U : \Gamma(U, \mathcal{O}_U^\times) \to H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\overline{k(X)})) = H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\bar{k}))$

$\cong H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

$\uparrow$

$\mu_{\hat{\mathbb{Z}}}(\bar{k}) \cong \mu_{\hat{\mathbb{Z}}}(\Pi_X)$

(1). κ_U: inj

non-interference

an splitting

concerning

"nal. gp portion"
of Θ-link

rom

ect

l. gp portion

$\hat{\mathbb{Z}}^\times$-indet.

Θ-ll

unit ${}^\dagger\mathcal{O}^\times \xrightarrow{\sim} {}^\ddagger\mathcal{O}^\times$

$\boxed{\hat{\mathbb{Z}}^\times\text{-indet}}$

cone

val. gp $\{\underline{q}^{j^2}\} \to \underline{q}$

gl. realif'd $\mathbb{R}_{\ge 0}$

portion"

$\sim \hat{\mathbb{Z}} = \mathrm{Hom}(I_x, I_x) \cong H^0(\Delta_{\bar{X}}, H^1(I_x, I_x)) = E_2^{0,1}$

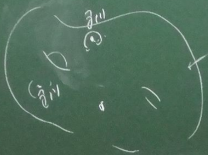
$\longrightarrow E_2^{2,0} = H^2(\Delta_{\bar{X}}, H^0(I_x, I_x))$

$\cong \mathrm{Hom}(\mu_{\hat{\mathbb{Z}}}(\Pi_X), I_x)$

$1 \longmapsto \mu_{\hat{\mathbb{Z}}}(\Pi_X) \xrightarrow{\sim} I_x$

(Cyc. Rig. Iner.)

cyclotomic rigidity
for inertia subgp



orient. $\hat{\mathbb{Z}}(1)$

$G_{\mathbb{Q}} \subset \widehat{GT}$

$\hat{\mathbb{Z}}^\times$

$H^2 \cong \hat{\mathbb{Z}}(-1)$

(2). (cf. [Cusp, Pr

∀ divisor D

reduc

$'(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\bar{k}))$

$H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

$_{\hat{\mathbb{Z}}}(\bar{k}) \overset{\sim}{=} \mu_{\hat{\mathbb{Z}}}(\Pi_X)$

$^0(I_x, I_x))$

$\mathrm{om}(\underset{\cup}{\mu_{\hat{\mathbb{Z}}}(\Pi_X)}, I_x)$

$_{\hat{\mathbb{Z}}}(\Pi_X) \xrightarrow{\sim} I_x$

rigidity

nertia subgp

(3). (cf. [Cusp, Prop 2.1 (i)])

$U = X \setminus S$, $S \subset X(k)$ fin. set

$\Pi_U \twoheadrightarrow \Pi_U^{\text{cusp-cont}}$

$\underset{\text{induces}}{\sim}$, $H^1(\Pi_U^{\text{cusp-cont}}, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \xrightarrow{\sim} H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

(4). (cf. [Cusp, Prop 1.4 (iii)]) $H^1(\Pi_X, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \supseteq (k^\times)^\wedge$

(Cyc. Rig. Iner.)

$\hat{\mathbb{Z}}$

$\wr\!\!\wr$

$\mathrm{Hom}_{\Pi_X}(I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

(5). (cf. [Cusp, Prop 2.1 (ii)])

restrictions to I_x $(x \in S)$

$$0 \to H^1(\Pi_X, H^0(\prod_{x \in S} I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X))) \to H^1(\Pi_U^{\text{cusp-cont}}, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} H^0(\Pi_X, H^1(I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X)))$$

(2). (cf. [Cusp, Prop 2.3 (ii)])

$\forall$ divisor D of $\deg = 0$ on X s.t. $\mathrm{Supp}(D) \subset X(k)$,

the section $s_D : G_k \to \Pi_J$ is equal to

(up to conj by Δ_X) the section

det'd by the origin $O \in J(k)$

$\Longleftrightarrow$ D is principal

X / k

J^d $\deg = d$

$J = J^0$-torsor

$X \to J^1$

$P \mapsto \mathcal{O}(P)$

$\Pi_X \to \Pi_{J^1}$

$x \in X(k)$ ⇝

$s_x : G_k \to \Pi_X \to \Pi_{J^1}$

$\Pi_{J^1} \times \cdots \times \Pi_{J^1} \to \Pi_{J^d}$

div D $\deg = d$ s.t. $\mathrm{Supp}(D) \subset X(k)$

⇝ $s_D : G_k \to \Pi_{J^d}$

$\bar{k} \supset \overset{\text{alg. closure}}{\bar{k}_{NF}} \supset \mathbb{Q}$

If $X_{\bar{k}}$ def'd / $\bar{k}_{NF}$, we say X is an NF-curve

X: NF-curve, pts of $X(\bar{k})$ (resp. rat. fcts on $X_{\bar{k}}$, $\overset{\bar{k} \subset \bar{k}(X)}{\text{const. rat. fcts}}$) which descend to $\bar{k}_{NF}$

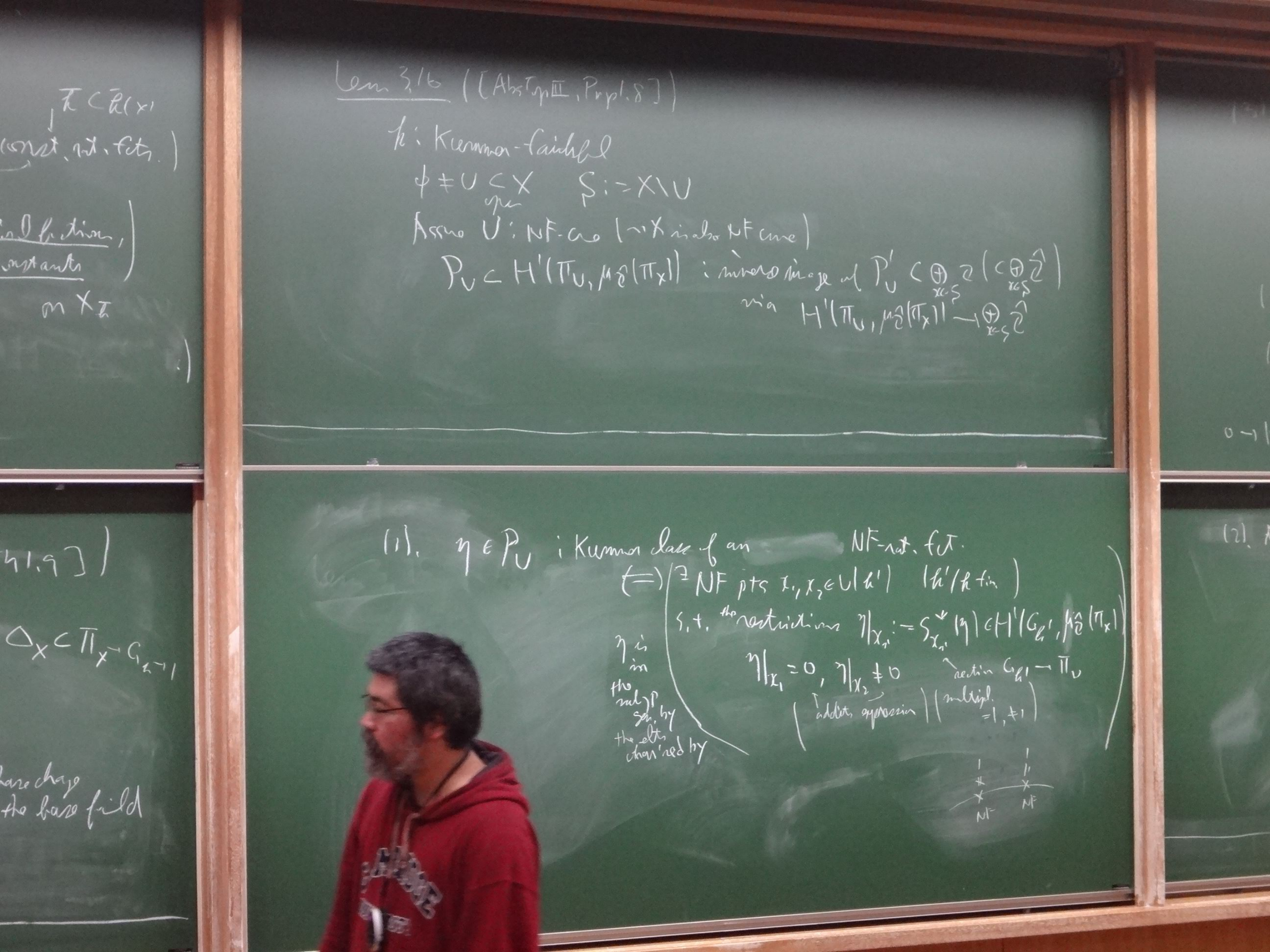
we call them NF-points (resp. NF-rational functions, NF-constants) on $X_{\bar{k}}$

$$\xrightarrow[(3),(4)]{\sim} \quad 1 \to (\bar{k}^\times)^\wedge \to H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} \hat{\mathbb{Z}}$$

(6): the image of $\Gamma(U, \mathcal{O}_U^\times)$ in $H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))/(\bar{k}^\times)^\wedge$ via κ_U is equal to the inverse image in $H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))/(\bar{k}^\times)^\wedge$ of the submodule $P'_X \subset \bigoplus_{x \in S} \mathbb{Z}$ $\left(\subset \bigoplus_{x \in S} \hat{\mathbb{Z}}\right)$ det'd by the principal divisors w/ support in S.

by cycl. rig.

1



Lem 3.16 ([AbsTopIII, Prop 1.8])

k: Kummer-faithful

$\phi \neq U \underset{\text{open}}{\subset} X \quad S := X \setminus U$

Assume U: NF-curve ($\to X$ is also NF curve)

$P_U \subset H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$: inverse image of $P'_U \subset \bigoplus_{x \in S} \mathbb{Z} \; (\subset \bigoplus_{x \in S} \hat{\mathbb{Z}})$

via $H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} \hat{\mathbb{Z}}$

(1). $\eta \in P_U$: Kummer class of an NF-rat. fct.

($\Longleftrightarrow$) ($\exists$ NF pts $x_1, x_2 \in U(k')$ (k'/k fin.)

s.t. the restrictions $\eta|_{x_i} := s_{x_i}^*(\eta) \in H^1(G_{k'}, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

$\eta|_{x_1} = 0, \ \eta|_{x_2} \neq 0$) — section $G_{k'} \to \Pi_U$

(additive expression) ((multipl. $=1, \neq 1$)

η is in the subgp gen. by the elts char'zed by

(3). (cf. [Cusp, Prop 2.1 (i)])

$U = X \setminus S$, $S \subset X(k)$ fin. set

$\Pi_U \twoheadrightarrow \Pi_U^{\text{cusp-cent}}$

$\underset{\text{induces}}{\sim}$ $H^1(\Pi_U^{\text{cusp-cent}}, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \hookrightarrow H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

(4). (cf. [Cusp, Prop 1.4 (ii)])

$H^1(\Pi_X, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \supseteq (k^\times)^\wedge$

(Cusp, Prop, Inter.)

$\hat{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Hom}_{\Pi_X}(I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

(5). (cf. [Cusp, Prop 2.1 (ii)])

restrictions to I_x ($x \in S$)

$$0 \to H^1(\Pi_X, H^0(\prod_{x\in S} I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X))) \to H^1(\Pi_U^{\text{cusp-cent}}, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x\in S} H^0(\Pi_X, H^1(I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X)))$$

(2). Assume that $\exists$ non-const. NF-rat. fcts in $\Gamma(U, \mathcal{O}_U^\times)$

$\eta \in P_U \cap H^1(G_k, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \supseteq (k^\times)^\wedge$

Kummer class of an NF-const. in $k^\times$

$\Longleftarrow$ $\exists$ a non-const NF-rat. fct $f \in \Gamma(U, \mathcal{O}_U^\times)$

& $\exists$ an NF-pt $x \in U(k')$ w/ k'/k fin.

s.t. $\kappa_U(f)|_x = \eta|_x$ in $H^1(G_{k'}, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

Th 3.17 (Mono-Anab. Reconstn. of NF-Portion [AbsTopIII, Th 1.9])

k : mlt-p-adic, X : str. Belyi type $\subset \overline{X}$

input $1 \to \Delta_X \subset \Pi_X \to G_k \to 1$

We can (gp th'ally) recon. $\overline{k}_{NF}(X)$, $\overline{k}_{NF}$ as following:

functorially

w.r.t. | open inj. hom. of ext'n of prof. gps
| hom. of ext'n of prof. gp arising from a base change of the base field

Step 1 By Belyi cuspidalization (Th 3.8)

$\underset{\text{gp th'ically}}{\widetilde{\quad}}$ the set of surj. $\{\Pi_U \twoheadrightarrow \Pi_X\}_U$

for open mlt-NF-curves $\phi \neq U \subset X$

& decomp. gps $D_x \subset \Pi_X$ of NF pts x

& inertia $I_x = D_x \cap \Delta_U$

Step 2 (Cycl. Rig. Iner.) (Prop 3.14)

$\underset{\text{gp th'ic}}{\widetilde{\quad}}$ $\underset{\uparrow \text{ Step 1}}{I_x} \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}(\Pi_X)$ $x \in X(k)$

Step 4

Step 3

i])

$\subset \Pi_X \to G_k \to 1$

map
ase field

Step 4 By the char'zation of non const. NF-rat. fcts & NF-const. (Lem 3.16 (1), (2))
in P_v ← Step 3

$\underset{\text{gp th'c}}{\widetilde{}}$ recon. subgp (via Kummer map κ_v's)

$$\overline{k}_{NF}^{\times} \subset \overline{k}_{NF}(X)^{\times} \subset \varinjlim_{U} H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$$

U: NF-cores of $\overline{X} \times_k k'$, k'/k fin.

Step 3

restr. $\overset{\text{Step 1}}{I_x} \rightsquigarrow H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to H^1(I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

cycl. rig (Step 2) $\rightsquigarrow 1 \to \ker \to H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} \hat{\mathbb{Z}}$

(Lem 3.15 (1))

By the char'zation of princ'l cuspidal div. (Lem 3.15(2) & decomp. gp in Step 1)

$$\underset{\text{gp th'c recon.}}{\rightsquigarrow} P_v \subset H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_U))$$

$(1), (2)$

$n\}$

Π mono

$\mathrm{Hom}(\Pi_1, \Pi_2)$ h:

mono Θ S^{Θ}

$S^{\mathrm{alg}}, S^{\Theta}$

$\overline{M}$

Θ $\boxplus$ $\longrightarrow \mathbb{F}^{\times}$

$\mathbb{F}_{\ell}^{\times}\{\pm 1\}$

$\Pi_S^{\times}$

Step 5

(a) $\overline{k}_{NF}(X)^{\times}$ $\leadsto$ Step 4

(b) ord_x's $\leadsto$ comp. at x of hom $H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \underset{\substack{\uparrow \\ \text{Step 3}}}{\longrightarrow} \bigoplus_{x \in S} \hat{\mathbb{Z}}$

(c) U_x $\leadsto$ $f(x) = 1$ $\leadsto$ rest. to decomp. D_x

Uchida's lemma Prop 3.12 $\rightarrow$ gp th'ic recon. additive str's on $\overline{k}_{NF}^{\times} \cup \{0\}$, $\overline{k}_{NF}(X)^{\times} \cup \{0\}$ //

OK.

Rem 3.17.1 k: NF, MLF $\leadsto$ input data only Π_X

Rem 3.17.2 $1 \to \Delta_{\eta_X} \to \Pi_{\eta_X} \to G_k \to 1$

$\uparrow$ gen. pt

similar constr. no need of (GC)

k: Kummer-faithful I_n (Step 1), Belyi's cusp'ization

$\underset{\text{recon.}}{\leadsto}$ $\overline{k}(X)$, $\overline{k}$

Rem 3.17.3 (slim of G_k for Kummer-faithful k)

use

k: Kummer-faithful $\Rightarrow$ G_k: slim cf. [AbsTopIII, Th 1.11] [, GC, Lem 15.8] //

Cor 3.19 (Mono-Anab. Recon. over MLF [AbsTopIII, Cor 1.10, Prop 3.2(i), Rem 3.2.1])

$k/\mathbb{Q}_p$ fin. X: hyperb. curve of str. Belyi type

prof. gp Π_X gp th'ically recon. the following

(1). the set of decomp. gps of all closed pts in X.

(2). $\overline{k}(X)$, $\overline{k}$ fields

(3). nat. isom. $\mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu^{K}_{\hat{\mathbb{Z}}}(\Pi_X) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, K(\overline{k}^{\times}_{NF}))$

(Cyc. Rig. LCFT) cycl. rig. via LCFT $K : \overline{k}^{\times}_{NF} \hookrightarrow \varinjlim_{U} H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

classical cycl. rig.

Def 3.18 $k/\mathbb{Q}_p$ fin. $\overline{k}$

$$\mu_{\mathbb{Q}/\mathbb{Z}}(G_k) := \varinjlim_{\substack{H \subset G_k \\ \text{open}}} (H^{ab})_{tor}, \quad \mu_{\hat{\mathbb{Z}}}(G_k) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_{\mathbb{Q}/\mathbb{Z}}(G_k))$$

Verlangerung

cyclotome of G_k

(1.10, Prop 3.2(ii), Rem 3.2.1])

e

X.

$\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, k(\bar{k}_{NF}^{\times}))$

$\bar{k}_{NF}^{\times} \hookrightarrow \prod_{v} H^1(\Pi_v, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

$= \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_{\mathbb{Q}/\mathbb{Z}}(G_k))$

<u>k</u>

gp th'c recon. k as completion of the field

$(H^1(G_k, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \cap \bar{k}_{NF}^{\times})^{\cup}\{0\}$

field str. on $\bar{k}_{NF}^{\times \cup}\{0\}$

w.r.t. val. det'd by the subring of $(H^1(G_k, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \cap \bar{k}_{NF}^{\times})^{\cup}\{0\}$ gen. by $\mathrm{Ker}(H^1(G_k, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \hat{\mathbb{Z}}) \cap \bar{k}_{NF}^{\times}$

indep. of the choice of $\mu_{\hat{\mathbb{Z}}}(G_k) \cong \mu_{\hat{\mathbb{Z}}}(\Pi_X)$

$\rightsquigarrow \bar{k}(X) := \bar{k} \otimes_{\bar{k}_{NF}} \bar{k}_{NF}(X)$.

Mod) (1). Cor 3.9

(2). Th 3.17 & Cor 2.4 $\rightsquigarrow \bar{k}_{NF}(X), \bar{k}_{NF}$.

Prop 2.1 (17) $\underset{\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, -)}{\overset{\text{gp th'c}}{\rightsquigarrow}}$ $H^2(G_k, \mu_{\hat{\mathbb{Z}}}(G_k)) \xrightarrow{\sim} \hat{\mathbb{Z}}$

cup prod.

$H^1(G_k, \mu_{\hat{\mathbb{Z}}}(G_k)) \xrightarrow{\sim} \mathrm{Hom}(H^1(G_k, \hat{\mathbb{Z}}), \hat{\mathbb{Z}}) \cong G_k^{ab}$

$G_k^{ab} \twoheadrightarrow G_k^{ab}/\mathrm{Im}(I_k) \cong \hat{\mathbb{Z}}$

$\underset{\text{gp th'c}}{\rightsquigarrow}$ $H^1(G_k, \mu_{\hat{\mathbb{Z}}}(G_k)) \twoheadrightarrow \hat{\mathbb{Z}}$

$\mu_{\hat{\mathbb{Z}}}(G_k) \cong \mu_{\hat{\mathbb{Z}}}(\Pi_X)$ up to $\hat{\mathbb{Z}}^{\times}$-mult.

(3). $\mu^k_{\mathbb{Q}/\mathbb{Z}}(\Pi_X) := \mu^k_{\hat{\mathbb{Z}}}(\Pi_X)\otimes\mathbb{Q}/\mathbb{Z}$

sp th'c $G^{ur} := \mathrm{Gal}(k^{ur}/k)$

Prop 2.1 (4a)   by the same way as Prop 2.1 (7),

$$H^2(G_k, \mu^k_{\mathbb{Q}/\mathbb{Z}}(\Pi_X)) \xrightarrow{\sim} H^2(G_k, k(\bar{k}^\times)) \xleftarrow{\sim} H^2(G^{ur}, k((k^{ur})^\times))$$

$$\xrightarrow{\sim} H^2(G^{ur}, \mathbb{Z}) \xleftarrow{\sim} H^1(G^{ur}, \mathbb{Q}/\mathbb{Z})$$

$$= \mathrm{Hom}(G^{ur}, \mathbb{Q}/\mathbb{Z}) \underset{\text{Frob. ch.}}{\xrightarrow{\sim}} \mathbb{Q}/\mathbb{Z}$$

$$\overset{\mathrm{Hom}(\mathbb{Q}/\mathbb{Z},-)}{\rightsquigarrow} H^2(G_k, \mu^k_{\hat{\mathbb{Z}}}(\Pi_X)) \xrightarrow{\sim} \hat{\mathbb{Z}} \rightsquigarrow \mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu^k_{\hat{\mathbb{Z}}}(\Pi_X)$$

$$H^2(G_k, \mu_{\hat{\mathbb{Z}}}(G_k))^{\wr} \text{ in (2)}$$

(by imposing the compat. cond. we get) //

Rem 3.19.2 ([AbsTop III, Prop 3.2, Prop 3.3])

$G_k \curvearrowright M$ top. monoid (resp. top. gp) $\cong \mathcal{O}^{\triangleright}_{\bar{k}}$ (resp. $\bar{k}^\times$)

compat. w/ G_k action

$\mu_{\hat{\mathbb{Z}}}(M) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, M^\times)$

$\mu_{\mathbb{Q}/\mathbb{Z}}(M) := \mu_{\hat{\mathbb{Z}}}(M)\otimes_{\hat{\mathbb{Z}}}\mathbb{Q}/\mathbb{Z}$

$M^{ur} := M^{\ker(G_k\to G^{ur})}$

We can take the generator of $M^{ur}/M^\times \cong \mathbb{N}$

(resp. gen. of $\underset{\cong\mathbb{Z}}{M^{ur}/M^\times}$ up to $\{\pm1\}$)

the set of decomp. gp's of all closed pts in X. the follo[illegible]

(2). $\overline{k}(X)$, $\overline{k}$ fields

(3). nat. isom. $\mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu^{k}_{\hat{\mathbb{Z}}}(\Pi_X) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, k(\overline{k}^{\times}_{NF}))$

(Cyc. Rig. LCFT) cycl. rig. via LCFT $\kappa : \overline{k}^{\times}_{NF} \hookrightarrow \varinjlim_{U} H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

classical cycl. rig.

$\rightsquigarrow H^2(G_k, \mu_{\mathbb{Q}/\mathbb{Z}}(M)) \xrightarrow{\sim} H^2(G_k, M^{gp}) \xleftarrow{\sim} H^2(G^{ur}, (M^{ur})^{gp})$

$\xrightarrow{\sim} H^2(G^{ur}, (M^{ur})^{gp}/(M^{ur})^{\times})$

$\xrightarrow{\sim} H^2(G^{ur}, \mathbb{Z})$

(*) $\xleftarrow{\sim} H^1(G^{ur}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G^{ur}, \mathbb{Q}/\mathbb{Z})$

$\xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$

can. def'd (resp. well-def. up to $\{\pm 1\}$)

$\rightsquigarrow \mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}(M)$ (Cyc. Rig. LCFT)

by the same Cor 3.19(3).

(resp. up to $\{\pm 1\}$) [illegible]

$\mathcal{O}_{\overline{k}} \setminus \{0\}$

!!

$\mathcal{O}^{\triangleright}_{\overline{k}}$ ⟳ rig.

$\overline{k}^{\times}$ $\rightsquigarrow$ rig. up to $\{\pm 1\}$

$\mathcal{O}^{\times}_{\overline{k}}$ $\rightsquigarrow$ rig. up to $\hat{\mathbb{Z}}^{\times}$

= no-rigidity

theta fct. mult. HA and theta value unnad.

log + indet. multirad.

Mod) (1). Cor 3.9

(2). Th 3.17 & C

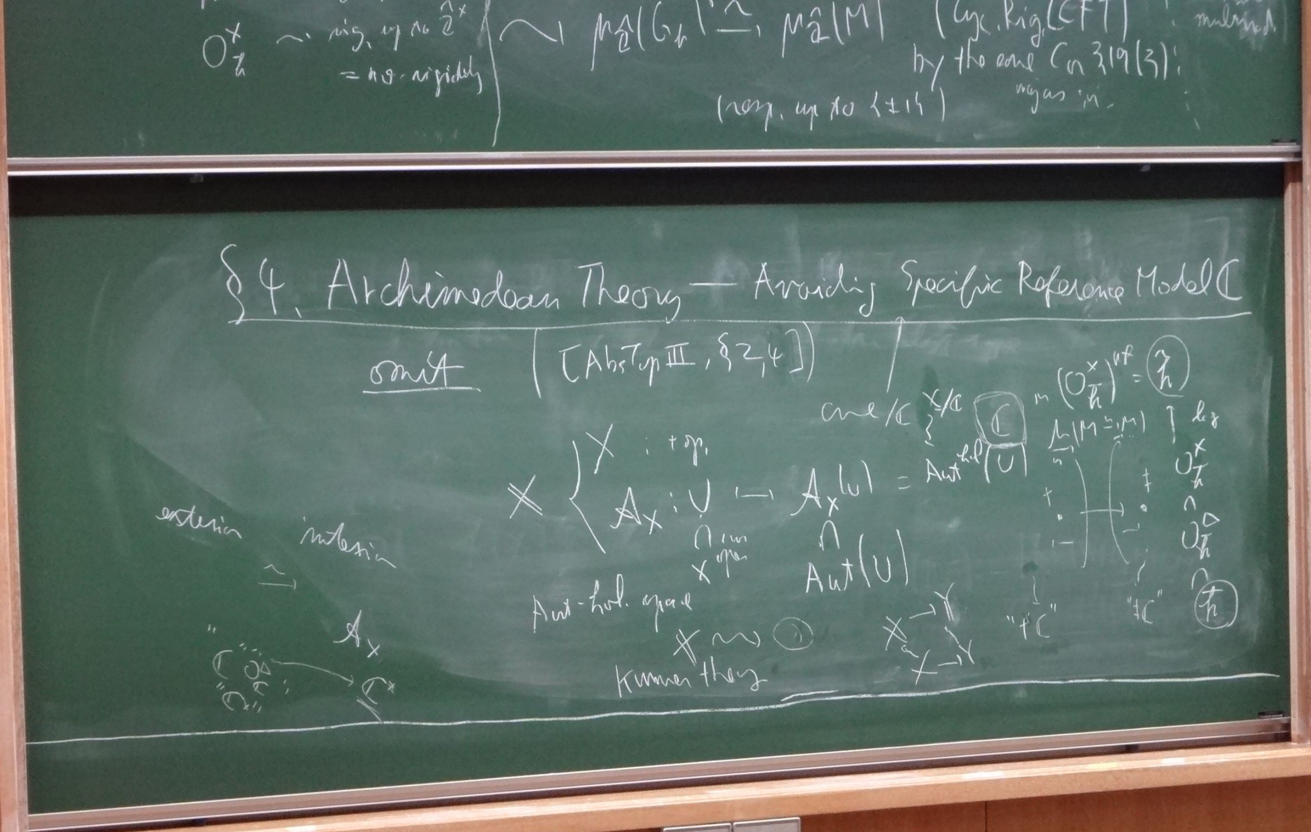
Prop 2.1

$\pm\mathcal{O}^{\times}/\mu \cong \pm\mathcal{O}^{\times}/\mu$

up to $\hat{\mathbb{Z}}^{\times}$


§4. Archimedean Theory — Avoiding Specific Reference Model C
EASY RISE
SET UP
CAMBRIDGE
UNIVERSITY


§4. Archimedean Theory — Avoiding Specific Reference Model ℂ
EASY RIDE
GET UP
CAMBRIDGE
UNIVERSITY



$\mathcal{O}^\times_{\bar{k}}$ ⤳ rig. up to $\hat{\mathbb{Z}}^\times$ = no rigidity

$\sim$ $\mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}(M)$ (Cyc. Rig. (CFT)) multirad.

by the same (in §1 9(3)); (resp. up to $\{\pm 1\}$) ... as in

§4. Archimedean Theory — Avoiding Specific Reference Model $\mathbb{C}$

omit ([AbsTopIII, §2, 4])

curve$/\mathbb{C}$ $X/\mathbb{C}$ $\boxed{\mathbb{C}}$ ⤳ $(\mathcal{O}^\times_{\bar{k}})^{\mathrm{rf}} = \bar{k}$

$\mathbb{X}\begin{cases} X : \text{top.} \\ \mathbb{A}_X : U \mapsto \mathbb{A}_X(U) = \mathrm{Aut}^{\mathrm{hol}}(U) \\ \quad\ \ U \subset X \text{ open},\quad \mathbb{A}_X(U) \subset \mathrm{Aut}(U) \end{cases}$

Aut-hol. space

$\mathbb{X}$ ⤳ ①

Kummer theory

$\Lambda(M \xrightarrow{\sim} M)$

"$\pm\mathbb{C}$" "$\pm\mathbb{C}$"

$\bar{k} \xrightarrow{\log} \mathcal{O}^\times_{\bar{k}} \subset \mathcal{O}^\triangleright_{\bar{k}} \subset \bar{k}$

$\mathbb{X} \to \mathbb{Y}$, $\mathbb{X}' \to \mathbb{Y}'$

exterior $\xrightarrow{\sim}$ interior

" $\mathbb{C}$ $\mathcal{O}^\triangleright$ " $\to$ $\mathbb{A}_X$ $\mathbb{C}^\times$ "$\mathcal{O}^\times$"

$)^{gp})$

$)^{\times})$

$(G^{ur}, \mathbb{Q}/\mathbb{Z})$

$\to \mathbb{Q}/\mathbb{Z}$

$\{\pm 1\})$

$\mathrm{Rig}(\mathbb{C}FT)$

one Cn 3.19(3)

mojas in

theta fct mult. HA and theta value unimod.

{log + index multirad.

X/k hyperbolic

$$\mathcal{O}^{\triangleright}_{\bar k} \supset \mathcal{O}^{\times}_{\bar k} \twoheadrightarrow \tilde{k} = (\mathcal{O}^{\triangleright}_{\tilde k})^{gp} := (\mathcal{O}^{\triangleright}_{k^\sim})^{gp} \cup \{0\} \leftarrow \mathcal{O}^{\triangleright}_{k^\sim}$$

$\curvearrowleft \Pi_X$ $\cap$ $k^\sim$

Def 5.1

$$\mathcal{I}_k := \frac{1}{p^*}\mathrm{Im}\{\mathcal{O}^{\times}_{k} \to (\mathcal{O}^{\times}_{\bar k})^{pf} = \tilde{k}\} \quad (G_k \curvearrowright \bar k)$$

$(\mathcal{O}^{\Pi_X}_{k^\sim} \curvearrowright)$

$$p^* = \begin{cases} p & p > 2 \\ p^2 & p = 2 \end{cases}$$

log-shell of k, mchi.

§5. Log-Volumes and Log-Shells

§5.1 non-Arch. Places

$k/\mathbb{Q}_p$ fin. $\subset \bar{k}$

$$\tilde{k} := (\mathcal{O}^{\times}_{\bar k})^{pf} := \varprojlim_n (\mathcal{O}^{\times}_{\bar k} \to \mathcal{O}^{\times}_{\bar k} \to \cdots)$$

$\begin{pmatrix} \uparrow \\ \mathcal{O}^{\times}_{\bar k} \end{pmatrix}$

p-adic log : $\log_{\bar k} : \tilde{k} \xrightarrow{\sim} \bar{k}$

ring str. input field str.

$\mathcal{O}^{\triangleright}_{k^\sim}$

$${}^\dagger\mathcal{O}^{\times}/\mu \cong {}^\ddagger\mathcal{O}^{\times}/\mu$$

μ in $\hat{\mathcal{O}}^{\times}$

Reference Model $\mathbb{C}$

$\mathbb{C} \cup$

$(\mathcal{O}^{\times}_{\bar k})^{pf} = \boxed{\tilde{k}}$

$\varprojlim_n (\mu \xrightarrow{n} \mu)$ $\uparrow$ log

$\mathcal{O}^{\times}_{\bar k} \cap \mathcal{O}^{\triangleright}_{\bar k} \cap \boxed{\bar{k}}$

"$^\dagger\mathbb{C}$" "$^\ddagger\mathbb{C}$"

(3). $\mu^{k}_{\mathbb{Q}/\mathbb{Z}}(\Pi_X)$

$\Pi \curvearrowright G^{ur} := G_k/I($

Prop 3.1 (4a)

$\mathcal{O}^{\times \Pi_X}_{k^\sim} \subset \mathcal{I}_k$ (normal upper bound)

$\vee$

log $\log(\mathcal{O}^{\times}_{k})$

{upper semi-compatibil

{log-Kummer corr

$\mathrm{End}(\mathbb{R}_{\mathrm{non}}(G)) \cong \mathbb{R}$ $\mathbb{R}_{\mathrm{non}}(G)$: $\mathbb{R}$-module

$\ni$

dist. elt $\mathbb{F}(G) \rightsquigarrow$ Frob. elt $\in \mathcal{O}^{\triangleright}_{\overline{k}}(G)^G / \mathcal{O}^{\times}_{\overline{k}}(G)^G$

$f_G \log p_G \in \mathbb{R}$ rebuild $\searrow$? $\mathbb{R}$-module

$\mathbb{F}(G) \in \mathbb{R}_{\mathrm{non}}(G) \leftarrow$ input

Step 4 $\mathbb{M}(\overline{k}^{\sim}(G)^G)$: the set of open cpt subsets of the top. additive gp $\overline{k}^{\sim}(G)^G$

recon. <u>local log-vol fct</u> $\mu^{\log}(G) : \mathbb{M}(\overline{k}^{\sim}(G)^G) \longrightarrow \mathbb{R}_{\mathrm{non}}(G)$

by the following properties

Step 2 $\mathcal{I}(G) := \frac{1}{p_G^*} \cdot \mathrm{Im}\left(\mathcal{O}^{\times}_{\overline{k}}(G)^G \longrightarrow \overline{k}^{\sim}(G) := \mathcal{O}^{\times}_{\overline{k}}(G)^{pf} \right)$

$\mathbb{Q} \hookrightarrow \mathrm{End}(\overline{k}^{\sim}(G))$

$$p_G^* := \begin{cases} p_G & p_G > 2 \\ p_G^2 & p_G = 2 \end{cases}$$

Step 3 $\mathbb{R}_{\mathrm{non}}(G) := \left(\overline{k}^{\times}(G) / \mathcal{O}^{\times}_{\overline{k}}(G) \right)^{\wedge}$

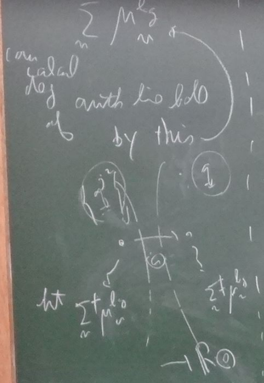
completion w.r.t. the order str. det'd by the image of $\mathcal{O}^{\triangleright}_{\overline{k}}(G) / \mathcal{O}^{\times}_{\overline{k}}(G)$

$\mathcal{O}^{\times}_{\hbar}(G)^{G}$

$\hbar^{n}(G)^{G}$

the Asp. additive gp

$) \longrightarrow R_{nm}(G)$



$q\mathcal{O} \subset q^{j^2}\mathcal{O}$

⸾

$\hbar^* \lesssim 0 + (\text{ind st})$

$\mu^{lg}(\theta) = 0$

$\mu^{lg}(m) = -\log q$

$\frac{1}{(:)}$

Pr

By the fol
alg

$\times_{\hbar}(G)^{\mu\dagger}$

(G)

nder str. dot'd

$/\mathcal{O}^{\times}_{\hbar}(G)$

(a). $A, B \in \mathcal{M}(\hbar^{n}(G)^{G})$ w/ $A \cap B = \phi$

(additivity)
$$\exp(\mu^{lg}(G)(A \cup B)) = \exp(\mu^{lg}(G)(A)) + \exp(\mu^{lg}(G)(B))$$

⸾ field str. on $R_{nm}(G)$

(b) $A \in \mathcal{M}(\hbar^{n}(G)^{G})$, $a \in \hbar^{n}(G)^{G}$, $\mu^{lg}(G)(A + a) = \mu^{lg}(G)(A)$

(t-transl. inv.)

(c)

(normalisation)
$$\mu^{lg}(G)(\mathcal{I}(G)) = \left(-1 - \frac{m_G}{f_G} + \varepsilon_G e_G f_G\right)|F(G)|, \quad \varepsilon_G = \begin{cases} 1 & p_G > 2 \\ 2 & p_G = 2 \end{cases}$$

§5.2 Arch. ← [AbsTopIII §5] ✓

§5.3 Globalize

$\begin{cases} 1, 3 & \text{non-Arch} \\ 2, 4 & \text{Arch} \\ 5 & \text{gl} \end{cases}$

$\mu^{\log}(\Theta) = 0$

$\mu^{\log}(m) = \underline{-\log q}$

$\frac{1}{(\ :\ )}$

$\mu^{\log}(G)(B))$

$^{\log}(G)(A)$

$= \begin{cases} 1 & p_G > 2 \\ 2 & p_G = 2 \end{cases}$

ch

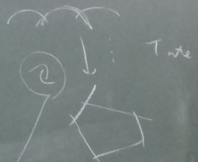
$Y \to X$ étale covering of rigid analytic spaces

is <u>tempered</u> $\overset{\Longleftrightarrow}{\text{def}}$ $\exists$

$$\begin{tikzcd} Z \arrow[r, "\text{top.}"] \arrow[d] & T \arrow[d, "\text{fin.}"] \\ Y \arrow[r] & X \end{tikzcd}$$

possibly infinite degree

dig. of fin. étale cov's



$\rightsquigarrow \pi_1^{\text{temp}}$

$\pi_1^{\text{temp}}(\mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, \infty\}) = \hat{\mathbb{Z}}$

e.g. E ell. curve $/\mathbb{C}_p$ $\quad \pi_1^{\text{temp}}(E) \cong \begin{cases} \hat{\mathbb{Z}} \times \hat{\mathbb{Z}} & |j|_p \leq 1 \\ \boxed{\mathbb{Z}} \overset{c}{\times} \hat{\mathbb{Z}} & |j|_p > 1 \end{cases}$

π_1^{temp} : neither discrete, nor profinite, nor locally compact,

but pro-discrete

§6. Preliminaries on Tempered Fundamental Groups

$/\mathbb{C}$ $\quad \pi_1^{\text{alg}}$ $\quad$ alg. var. $\rightsquigarrow$ <u>cpx mfd</u>

π_1^{top}

prof. $\quad$ disc.

alg. var. $\rightsquigarrow$ <u>rigid var.</u>

$/\mathbb{Q}_p$ $\quad \pi_1^{\text{alg}}$

no good $\begin{cases} \pi_1^{\text{top}} & \pi_1^{\text{top}}(\mathbb{P}^1_{\mathbb{C}_p} \setminus (N \text{ pts})) = \{1\} & \text{too small} \\ \pi^{\text{ét}} & \pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}) \to SL_2(\mathbb{Q}_p) & \text{too big} \end{cases}$

Gross-Hopkins

André $\rightsquigarrow$ tempered covering / fund. gp

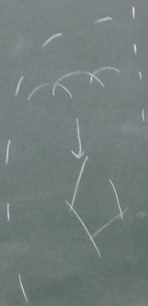
Cor 2.4 $\Rightarrow$ <u>Lem 6.2</u>

X: hyperbolic/k $\quad \Pi_X^{\text{tmp}} \underset{\text{gp thic}}{\rightsquigarrow} \Delta_X^{\text{tmp}}$

(☺ $(\Pi^{\text{tmp}})^\wedge = \Pi$)

$\overline{k} \supset K /^{\text{fin.}} \mathbb{Q}_p$

$\overline{k} \supset k$ vs. l.c.d



[IUTchII]

HA and Π^{tmp}/k_v

$\updownarrow$

labelling by <u>global data</u>

~~tmp~~ $\hat{\Pi}/K$

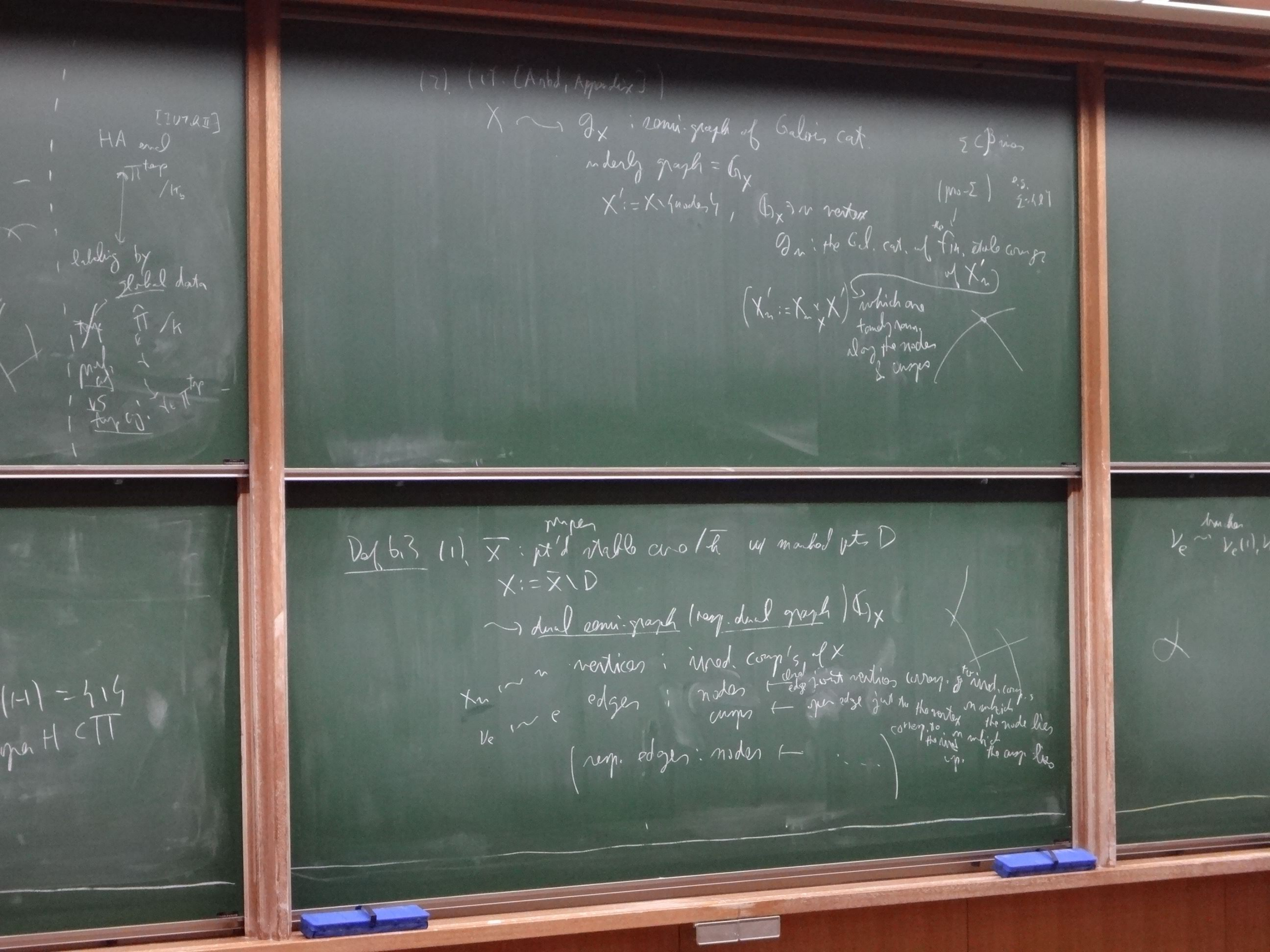
<u>prof. conj.</u> vs <u>temp. conj.</u> $\quad$ to Π^{tmp}

<u>Def 6.1</u> ([SemiAnbd, Def 3.1 (i), Def 3.4])

(1). temp. gp $\Pi \overset{?}{=} \varprojlim_{\text{surj's}}$ (countable disc. top. gp)

$\nwarrow$ <u>tempered gp</u>

(2). Π : tempered gp. Π : <u>temp-slim</u> $\overset{\text{def}}{\Leftrightarrow} Z_\Pi(H) = \{1\}$ $\forall$ open $H \subset \Pi$



[IUTch II]

HA and

Π^{top} $/k_0$

labeling by
global data

~~top~~ $\hat{\Pi}/k$

mod. vs top cj.

$\forall t\ \Pi^{\mathrm{top}}$

$(1-1) = \{1\}$

open $H \subset \Pi$

(2). (cf. [Anbd, Appendix])

$X \rightsquigarrow \mathcal{G}_X$: semi-graph of Galois cat.

underlying graph $= \mathbb{G}_X$

$X' := X \setminus \{\text{nodes}\}$, $\mathbb{G}_X \ni v$ vertex

$\mathcal{G}_v$: the Gal. cat. of the fin. étale covrgs of X'_v which are tamely ram. along the nodes & cusps

$(X'_v := X_v \times_X X')$

$\Sigma \subset \mathfrak{P}$ primes

(pro-Σ) e.g. $\Sigma = \{l\}$

Def. 6.3 (1). $\overline{X}$: pt'd proper stable crv $/\overline{k}$ w/ marked pts D

$X := \overline{X} \setminus D$

$\rightsquigarrow$ dual semi-graph (resp. dual graph) $\mathbb{G}_X$

$X_v \leftarrow\!\!\sim v$ vertices : irred. comp's of X

$V_e \leftarrow\!\!\sim e$ edges : nodes $\longmapsto$ closed edge joint vertices corresp. to the irred. comp.'s on which the node lies

cusps $\longmapsto$ open edge just to the vertex corresp. to the irred. cp. on which the cusp lies

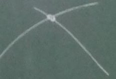
(resp. edges : nodes $\longmapsto$ ....)

$V_e \rightsquigarrow$ branches $V_e(1)$, V

$\Sigma \subset$ Primes

(mo-Σ) e.g. $\Sigma = \{l\}$

fin. étale cov.'s of X'_v



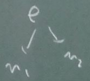
open edge e_x

x : cusp

X'_x : sch. th'c intersection of the completion of $\overline{X}$ at x w/ X'

$\wr$ non. can. Spec $\overline{k}((t))$

$\mathcal{G}_{e_x}$: Gal cat. of the fin. ét (mo-Σ) cov.'s of X'_x which are tame ram. along the cusp.

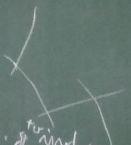


$\rightsquigarrow$ pull-back natural functors $\mathcal{G}_{v_1} \to \mathcal{G}_e \leftarrow \mathcal{G}_{v_2}$

open edge $e \to v$

$\mathcal{G}_v \to \mathcal{G}_e$

$\mathcal{G}_X := \{\mathcal{G}_v ; \mathcal{G}_e ; \mathcal{G}_v \to \mathcal{G}_e\}$



g. $\mathcal{G}^{tri}$ ind. comp.'s

vertex in which the node lies

irred. in which comp. the cusp lies

$v_e \overset{\text{branches}}{\rightsquigarrow} v_e(1), v_e(2)$

$X'_{v_e(i)}$: scheme th'c intersection of the completion of the bra.h $v_e(i)$ at the node v_e w/ X'

$(i = 1, 2)$ $\leftarrow$ non.can. Spec $\overline{k}((t))$

fix $\overline{k}$-isom $X'_{v_e(1)} \cong X'_{v_e(2)}$ $\leftarrow$ identify $\rightsquigarrow X'_e$

$\mathcal{G}_e$: the Gal. cat. of the fin. ét (mo-Σ) cov.'s of X'_e which are tamely ram. along the node

(4). (cf. [SemiAnbd, before Def 3.4 & Def 3.5])

Given $\mathcal{G} = \{\mathcal{G}_v; \mathcal{G}_e; \mathcal{G}_v \to \mathcal{G}_e\}$

$\rightsquigarrow$ cat $\mathcal{B}^{(\mathrm{cov})}(\mathcal{G})$: **Obj** data $\{S_v, \phi_e\}_{v,e}$ $\amalg$

Mor evident manner.

S_v : for a countable coproduct of conn. obj's of $\mathcal{G}_v$

$\phi_e : e(1)^* S_{v_1} \xrightarrow{\sim} e(2)^* S_{v_2}$ isom. in the cat. of fnl cnt. coprod. of conn. obj's of $\mathcal{G}_e$

(G: countable semi-graph $\rightsquigarrow \mathcal{B}^{(\mathrm{cov})}(\mathcal{G})$)

(3) [cf. [SemiAnbd, Def 2.1]]

Given $\mathcal{G} = \{\mathcal{G}_v; \mathcal{G}_e; \mathcal{G}_v \to \mathcal{G}_e\}$ / G

$\rightsquigarrow$ cat $\mathcal{B}(\mathcal{G})$

Obj data $\{S_v, \phi_e\}_{v,e}$

vertices edges

$S_v \in \mathrm{Ob}(\mathcal{G}_v)$

$\phi_e : e(1)^* S_{v_1} \xrightarrow{\sim} e(2)^* S_{v_2}$

branch of e

$e(i)^* : \mathcal{G}_{v_i} \to \mathcal{G}_e$

~~$\mathcal{G}$~~ $\pi_1(X,Y)/(X,Y)$

$\uparrow$ $X' \to X^e$, $Y' \to Y^e$

$\pi_1(X',Y')/(X',Y')$

Hom evident manner.

$\mathcal{B}(\mathcal{G})$: Galois cat. $\mathcal{G} = \mathcal{G}_X$ in (2) $\rightsquigarrow \pi_1(\mathcal{B}(\mathcal{G}))$ admissible fund. gp

II

i.e.

countable coprod. of

conn. objs. of $\mathcal{G}_v$

$S_{v_1} \xrightarrow{\sim} \rho(2)^* S_{v_2}$

in the cat. of

(countable coprod. of

conn. objs. of $\mathcal{G}_e$

-pendi

(2). (cf. [Abstracts, Appendix])

$X \rightsquigarrow \mathcal{G}_X$: semi-graph of Galois cat.

$\Sigma \subset \mathfrak{P}$rimes

underlying graph $= \mathbb{G}_X$

(pro-Σ) e.g. $\Sigma = \{\ell\}$

$X' := X \setminus \{\text{nodes}\}$, $\mathbb{G}_X \ni v$ vertex

$\mathcal{G}_v$: the Gal. cat. of the fin. étale coverings of X'_v

$(X'_v := X_v \times_X X')$ which are tamely ramified along the nodes & cusps

$(\mathcal{B}(\mathcal{G}) \subset)\ \mathcal{B}^{\text{temp}}(\mathcal{G}) \subset \mathcal{B}^{\text{cov}}(\mathcal{G})$

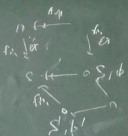
↑ full subcat.

obj. $\{S_v, \phi_e\}_{v,e}$ s.t.

$\exists \{S'_v, \phi'_e\}_{v,e} \in \mathrm{Obj}(\mathcal{B}(\mathcal{G}))$

$\forall$ vertex or edge c restriction of $\{S'_v, \phi'_e\}$ to $\mathcal{G}_c$ splits the restriction of $\{S_v, \phi_e\}$ to $\mathcal{G}_c$

i.e. the fiber product of S'_v (resp. ϕ'_e) w/ S_v (resp. ϕ_e) over the term. obj. (resp. id. morph. of term. obj.) in the cat. of (= countable coprod. of conn. objs. of $\mathcal{G}_v$ (resp. $\mathcal{G}_e$))



is

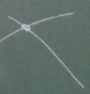
Rem G

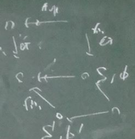
$\Sigma \subset \beta$ vias

$\Sigma)$ e.g. $\Sigma = \{\ell\}$

stable curve

$X'_n)$





$, \phi'_e\}$

splits

$\{\Sigma_v, \phi_e\}$ to $\mathcal{G}_c$

Σ_v (resp. ϕ_e)

$\mathcal{G}_v$ (resp. $\mathcal{G}_e$)

$$\rightsquigarrow \cdots \leftarrow \mathcal{B}^{\mathrm{top}}(\mathcal{G}_i) \leftarrow \cdots \leftarrow \mathcal{B}^{\mathrm{top}}(\mathcal{G})$$

compatible w/ actions of $\Delta_X^{\mathrm{top}}/N_i$

$$\rightsquigarrow \Delta_X^{\mathrm{top}} \twoheadrightarrow \cdots \twoheadrightarrow \pi_1(\mathcal{B}^{\mathrm{top}}(\mathcal{G}_i)) \overset{\mathrm{out}}{\rtimes} (\Delta_X^{\mathrm{top}}/N_i)$$

$$\twoheadrightarrow \cdots \twoheadrightarrow \pi_1^{\mathrm{top}}(\mathcal{G})$$

$$\rightsquigarrow \Delta_X^{\mathrm{top}} \cong \varprojlim_i \left(\pi_1(\mathcal{B}^{\mathrm{top}}(\mathcal{G}_i)) \overset{\mathrm{out}}{\rtimes} (\Delta_X^{\mathrm{top}}/N_i) \right)$$

$\circledast$

is isom. to the coprod. of a countable number of copies of Σ'_v (resp. ϕ'_e)

$\rightsquigarrow \pi_1(\mathcal{B}^{\mathrm{top}}(\mathcal{G}))$

Rem. 6.3.1 (cf. [SemiAnbd, Ex. 3.10])

X : smooth curve w/ marked pts $/\overline{K}$

special fiber of the stable model of X

$\rightsquigarrow \mathcal{G}$: semi-graph of G.l. cat's

$\Delta_X^{\mathrm{top}} := \pi_1^{\mathrm{top}}(X)$

$\cup N_i \supset \cdots)$ exhaustive seq. of open char. subgp of fin. index.

$\updownarrow$

$X_i \twoheadrightarrow X$

$\rightsquigarrow \mathcal{G}_i \curvearrowleft \Delta_X^{\mathrm{top}}/N_i$ faithful

$\pi_1^{tp}(\mathcal{G}) := \pi_1(\mathcal{B}^{tp}(\mathcal{G}))$

Prop 6.5 (cf. [SemiAnbd, Th 3.7])

(1). max. cpt subgps of $\pi_1^{tp}(\mathcal{G})$ up to conj. $\xleftrightarrow{1:1}$ verticial subgps of $\pi_1^{tp}(\mathcal{G})$ $\xleftrightarrow{1:1}$ vertex

$\uparrow$ image of $\pi_1(\mathcal{G}_v)$ for v: vertex

(2). $\forall$ edge-like subgp in $\pi_1^{tp}(\mathcal{G})$ $\xleftrightarrow{1:1}$ edge

$\uparrow$ image of $\pi_1(\mathcal{G}_e)$ for e: edge

closed

is contained in precisely 2 verticial subgps

((3). $\exists$ gp th'c char. for cusp. decp. gp $\subset \pi_1^{tp}(\mathcal{G})$

$\rightsquigarrow$ In particular gp th'c can recover $\mathbb{G}_X$)

⊛ & prime-Σ $\Sigma \not\ni p$ $\Rightarrow \pi_1^{adm}(\underline{\text{special}}) \cong \pi_1^{\Sigma}(X_{\bar{k}}) \rightsquigarrow \pi_1(X_{\bar{k}}) \underset{\text{gp th'c}}{\rightsquigarrow}$ graph of special fiber

Prop 6.4 (cf. [SemiAnbd, Prop 2.6, Cor 2.7, Prop 3.6])

$\pi_1(\mathcal{B}^{tp}(\mathcal{G}_{\cdot}))$: temp-slim

$\underset{⊛}{\rightsquigarrow} \Delta_X^{tp}$: temp-slim $\rightsquigarrow \Pi_X^{tp}$: temp-slim

Frobenioid

Fr'd $\boxed{εἶδος}$

ὕλη

ἄνδρωπος

$\Pi_X^{tp}[\mu_N]$

$\cup$

μ_N

(g)

$= \pi_1(B^{top}(g))$

ertex

$\pi_1^{top}(g)$

: vertex

l subgps

graph of
pecial fiber

enioid

εἶδος

ἰδη

νδρωπος

$\Lambda \rightsquigarrow v \rightsquigarrow v^{\gamma}$

$\gamma\Lambda\gamma^{-1} \rightsquigarrow v'$ $\gamma\Lambda\gamma^{-1}$

$\Pi_{v'} \cap \Pi_{v^{\gamma}} \neq \{1\}$

$\Rightarrow v' = v^{\gamma}$ or $v' — v^{\gamma}$

$\Rightarrow \gamma \in \Pi^{tp}$

By applying this to various coverings

Th 6.6 (Prof. Conj. vs Top. Conj [IUTchI, Prp 2.4, Cor 2.5])

$\Sigma = \{l\}$
$l \neq p$

$\gamma \in \hat{\Pi}_X$ s.t. $\gamma \Lambda \gamma^{-1} \subset \Delta_X^{top} \Rightarrow \gamma \in \Pi_X^{top}$

s.t. cpt sub $\subset \Delta_X^{top}$ Π_X^{top}

Cor 6.7 $\Pi_X^{top} \rightsquigarrow$ gp thic rem cmp. dom. gps s.t. [illegible] in G_k open

Π

Cor 2.9, Rem 2.9.1, Th 6.6 ← prof. conj. & Th 6.6